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GAS FLOWS WITH SHOCK WAVES WHICH DIVERGE FROM AN AXIS OR CENTRE OF SYMMETRY WITH FINITE VELOCITY†

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Cylindrically and spherically symmetric steady flows of an ideal gas are investigated. The Cauchy problem with data for $\zeta = 0$ is considered in the space of independent variables $\zeta = t/r$, $\chi = r$. The existence and uniqueness of the analytic solution of this problem is proved. The Cauchy problem with initial data on different surfaces is considered in the space of the t, r variables: for $r = 0$ the zero velocity of the gas is specified, and the Hugoniot conditions are satisfied on the unknown shock wave front, which diverges with finite velocity from an axis or centre of symmetry. The existence and uniqueness of the analytic solution of this problem is also proved, and the law of motion of the diverging shock wave is determined uniquely. Two problems on the focusing of a gas and on its subsequent reflection with a finite velocity of the shock wave are solved, namely, (1) the compression wave due to smooth motion of a piston in a gas at rest is focused, and (2) the rarefaction wave that arises when a one-dimensional cavity collapses is focused. The solutions of these problems represent an extension of Sedov's self-similar solutions to the case of two independent variables [1–3]. Moreover, the solution of the second problem extends the mathematical investigation of the process of one-dimensional cavity collapse [4, 5]. Copyright © 1996 Elsevier Science Ltd.

In the class of self-similar flows, which depend on one independent variable $\lambda = r/t$, solutions are known [1–3] which describe the focusing of a compression wave due to the smooth motion of a piston, in a uniform gas at rest, and which also describe [1] the focusing in a vacuum of a rarefaction wave with finite velocity. After focusing of the weak discontinuity or the free boundary, a reflected shock wave diverges with finite constant velocity from a centre or axis of symmetry, after which the compressed gas is at rest. Using the characteristic Cauchy problem [6] in a certain neighbourhood of the point $(t = t_0, r = r_0)$ $r_0 > 0$, the self-similar problems of a piston moving smoothly into a gas from the point $r = r_0$ [7] and the problem of the collapse of a one-dimensional cavity [4, 5] were solved. In the class of self-similar flows, which depend on a single variable $\xi = t/r^k$, $1 < k < 2$, flows with a shock wave reflected from the point $r = 0$ were constructed in [8, 9], the velocity of these flows being variable and equal to infinity at the point $r = 0$. The existence and uniqueness of piecewise-analytic solutions in the problem of the reflection of a three-dimensional shock wave from a curvilinear wall and in the problem of the interaction of curvilinear shock wave fronts were proved in [10, 11]. In these two problems it is necessary to investigate a Cauchy problem with initial data, specified simultaneously on different surfaces [12–14].

1. FORMULATION OF THE PROBLEMS

We will consider a system of equations of gas dynamics [15, 16] for an ideal polytropic gas with equation of state $p = A^2(S)\rho^\gamma/\gamma$, where p is the pressure, S is the entropy (henceforth we will denote the function $A(S)$ by s), ρ is the density, and $\gamma = \text{const} > 1$ is the polytropy index of the gas. We will investigate cylindrically symmetric ($\nu = 1$) or spherically symmetric ($\nu = 2$) flows, which depend on the time t and the distance $r = (x_1^2 + x_2^2 + \ldots + x_{\nu+1}^2)^{1/2}$ (x_1, x_2, x_3 are spatial coordinates). We will take $\mathbf{U} = (\sigma, u, s)$ ($\sigma = \rho^{(\gamma-1)/2}$, and u is the velocity of the gas) as the required functions $\mathbf{U} = \mathbf{U}(t, r)$. The velocity of sound in the gas is then given by the relation $c = \sigma s$, and the system of equations of gas dynamics has the form

$$\sigma_t + u\sigma_r + \frac{\gamma-1}{2}\sigma\left(u_r + \nu\frac{u}{r}\right) = 0$$

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$$u_t + \frac{2}{\gamma-1}\sigma s^2 \sigma_r + uu_r + \frac{2}{\gamma}\sigma^2 s s_r = 0 \tag{1.1}$$

$$s_t + us_r = 0$$

We will seek piecewise-analytic solutions of system (1.1) for the following two problem: (1) the smooth motion of a piston in a gas, which generates a focusing compression wave, and (2) the collapse of a one-dimensional cavity. The configuration of the corresponding flows in the plane of the variables t, r is shown in Fig. 1.

In the first problem, at the instant $t = t_0$ when $0 \leqslant r \leqslant r_0$ a uniform gas is at rest and an impenetrable piston begins to move smoothly into it from the point A ($t = t_0$, $r = r_0$) (the curve AB in Fig. 1 is the trajectory of the piston). A sound characteristic (the straight line AO in Fig. 1) begins to propagate with constant velocity c_0 in the uniform gas at rest in the region Ω_0 (c_0 is the velocity of sound in the gas in the region Ω_0). This straight line separates the region of the compression wave Ω, from the region at rest Ω_0. The instant when the characteristic AO is focused is taken as $t = 0$. For an analytic law of motion of the piston in a certain neighbourhood of the point A in the region Ω_1, there is a unique analytic solution of the problem of the piston [6, 7], which corresponds to isentropic flow. Outside this neighbourhood, in the region Ω_1, singularities of the gradient catastrophe type may occur.

It the law of motion of the piston is chosen in a special form, the flow in the region Ω_1, will be self-similar. More accurately [1–3], for the system of ordinary differential equations which describes self-similar flows a specific integral curve is constructed which passes through the corresponding singular points. A specific compression wave can thereby be chosen in the region Ω_1. From this one can uniquely establish the curves AO, AB and OC—the trajectory of motion of the reflected shock wave (see Fig. 1). For these self-similar flows the trajectory OC will be a straight line, while in the region Ω_2 between the reflected shock wave OC and the $r = 0$ axis the compressed gas will again be at rest and is uniform. In the region Ω_1 the parameters of the gas are constant along the straight lines λ = const, including $\sigma(0, r)$ = const > 0, $u(0, r)$ = const < 0. It is clear that arbitrary profiles of the gas-dynamic parameters are not specified by the self-similar flows $\mathbf{U} = \mathbf{U}(\lambda)$ at the instant $t = 0$

$$\sigma(0, r) = \sigma_0(r), \quad \sigma_0(0) > 0, \quad u(0, r) = u_0(r), \quad u_0(0) < 0$$

If we assume that for arbitrary $\sigma_0(r)$, $u_0(0)$ in the region Ω_1 when $t \geqslant 0$ there is a solution of system (1.1), the reflected shock wave OC will not be a straight line, and for the flow of gas in the region Ω_2, σ, u and s will not be constants.

In the second problem (on the collapse of a one-dimensional cavity [4, 5]) at the instant $t = t_0$ when $0 \leqslant r \leqslant r_0$ (Fig. 1) there is a vacuum, and when $r \geqslant r_0$ the distributions of the gas-dynamic parameters are specified so that $c(t_0, r_0) > 0$. The quantity $s(t_0, r)$ may then also not be constant. At the instant $t = t_0$ the wall $r = r_0$ is instantaneously removed and when $t > t_0$ leakage of the gas into the vacuum begins in the direction of an axis or centre of symmetry.

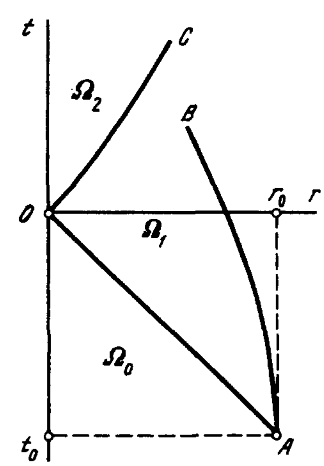

Fig. 1.

It was established in [4, 5] that in a certain neighbourhood of the point A there is a unique solution of the problem of the decay of such a discontinuity, and this solution is analytic at a certain time when $t > t_0$. For $\gamma < 3$ it was proved that the free boundary, which separates the region of the vacuum Ω_0 from the region of the focused rarefaction wave Ω_1 (Fig. 1), moves after a certain time with constant velocity equal to $u(t_0, r_0) - 2c(t_0, r_0)/\gamma - 1)$. Moreover, it was proved that when $\gamma \leqslant \gamma_* = 1 + 2/(\nu + 1)$ the free boundary, up to the instant when focusing occurs, has this constant velocity of motion. Further, after the instant $t = 0$ the instant when the free boundary is focused at the centre or axis of symmetry is chosen. In Fig. 1 the straight line AO corresponds to the free boundary.

The self-similar solutions $\mathbf{U}(\lambda)$ also describe [1] the focusing of the rarefaction wave on which the velocity of motion of the free boundary is constant. The distributions of the gas-dynamic parameters at a fixed instant of time are special and uniquely defined by the choice of the specific self-similar flow $\mathbf{U}(\lambda)$. Since, in all the flows mentioned at the instant when the rarefaction wave becomes focused, the velocity of the gas at the point $r = 0$ is finite and strictly negative, we can assume that when $t \geqslant 0$ the shock wave OC is reflected from the centre or axis of symmetry with finite velocity (Fig. 1); this will separate the flow in region Ω_1 from the flow in region Ω_2 between OC and the $r = 0$ axis. In general the flows in Ω_1 and Ω_2 are non-isentropic.

The purpose of this paper is as follows. Initially, starting from the arbitrary initial conditions $\mathbf{U}(0, r) = \mathbf{U}_0(r)$

$$\begin{aligned} \sigma(0,r) &= \sigma_0(r), \quad \sigma_0(0) > 0 \\ u(0,r) &= u_0(r), \quad u_0(0) < 0 \\ s(0,r) &= s_0(r), \quad s_0(0) < 0 \end{aligned} \tag{1.2}$$

it is necessary to construct a solution of system (1.1) in the region Ω_1 and to connect it with the problem of the focusing of either the compression wave (the first problem) or the rarefaction wave (the second problem). Then, in the region Ω_2 we need to construct another solution of system (1.1) for which $u(t, 0) = 0$. Simultaneously with the construction of the solution in Ω_2 we must construct the unknown shock wave OC, on which the flows constructed in Ω_1 and required in Ω_2 are related by the Hugoniot relations [15, 16].

2. CONSTRUCTION OF THE SOLUTIONS IN THE REGION Ω_1

If the data (1.2) in a certain neighbourhood of the point $r = 0$ are analytic functions, the solution of the Cauchy problem (1.1), (1.2) is uniquely constructed in the form of a formal power series

$$\mathbf{U}(t,r) = \sum_{k=0}^{\infty} \frac{\mathbf{U}_k(r) t^k}{k!}$$

Here it can be established by induction over k that $\mathbf{U}_k(r) = \mathbf{U}_{k0}(r)/r^k \geqslant 1$. The functions $\mathbf{U}_{k0}(r)$ are analytic in the same neighbourhood of the point $r = 0$ as the functions $\mathbf{U}_0(r)$ from (1.2). Consequently, the solution of problem (1.1), (1.2) can be represented uniquely in the form of a formal series

$$\mathbf{U}(t,r) = \sum_{k=0}^{\infty} \mathbf{U}_{k0}(r) \frac{(t/r)^k}{k!} \tag{2.1}$$

where $\mathbf{U}_{00}(r) = \mathbf{U}_0(r)$. In order to investigate the region of convergence (and, consequently, the region of applicability) of series (2.1), we make the following change of variables in (1.1)

$$\zeta = t/r, \quad \chi = r \tag{2.2}$$

with Jacobian $J = 1/r$. The change (2.2) is degenerate when $r = 0$ (to investigate different features of the solutions of the system of equations of gas dynamics degenerate changes of variables are often employed—see, for example, [4, 5, 7, 17, 18]).

System (1.1) in ζ, χ variables can be written in the form

$$(1 - \zeta u)\sigma_\zeta - \frac{\gamma - 1}{2}\zeta\sigma u_\zeta + \chi\left[u\sigma_\chi + \frac{\gamma - 1}{2}\sigma u_\chi\right] + \nu\frac{\gamma - 1}{2}\sigma u = 0$$

$$-\frac{2}{\gamma-1}\zeta\sigma s^2\sigma_\zeta+(1-\zeta u)u_\zeta-\frac{2}{\gamma}\zeta\sigma^2 ss_\zeta+\chi\left[\frac{2}{\gamma-1}\sigma s^2\sigma_\chi+uu_\chi+\frac{2}{\gamma}\sigma^2 ss_\chi\right]=0 \tag{2.3}$$

$$(1-\zeta u)s_\zeta+\chi us_\chi=0$$

If we put $\partial/\partial\chi = 0$ in (2.3), we obtain a system of ordinary differential equations equivalent to the corresponding system of ordinary differential equations from [1–3] and which describe self-similar flows which depend only on ζ.

It follows from the form of (2.2) that the instant $t = 0$ corresponds to the line $\zeta = 0$ and the initial conditions (1.2) become the following initial conditions

$$\mathbf{U}(\zeta,\chi)|_{\zeta=0}=\mathbf{U}_0(\chi) \tag{2.4}$$

Theorem 1. If $\mathbf{U}_0(\chi)$ are functions that are analytic in a certain neighbourhood of the point $\chi = 0$, the Cauchy problem (2.3), (2.4) has the following unique analytic solution in a certain neighbourhood of the point $\chi = 0$

$$\mathbf{U}(\zeta,\chi)=\sum_{k=0}^{\infty}\mathbf{U}_{k1}(\chi)\frac{\zeta^k}{k!},\quad \mathbf{U}_{01}(\chi)=\mathbf{U}_0(\chi) \tag{2.5}$$

Here, if $s_0(\chi) = s_0(0) = \text{const}$ we have $s(\zeta, \chi) = s_0(0)$.

This theorem is a corollary of the Cauchy–Kovalevskii theorem.

In order to obtain the boundary points of the region of convergence of series (2.5) on the $O\zeta$ axis, we will represent the solution of problem (2.3), (2.4) in the form

$$\mathbf{U}(\zeta,\chi)=\sum_{k=0}^{\infty}\mathbf{U}_{k2}(\zeta)\frac{\chi^k}{k!}. \tag{2.6}$$

Then $\mathbf{U}_{02}(\chi)$ is a solution of the Cauchy problem, which is obtained if we put $\chi = 0$ in problem (2.3), (2.4) (problem A).

Because of the degeneracy of conversion (2.2), system (2.3) takes the same form as when $\partial/\partial\chi = 0$ when $\chi = 0$, i.e. the system of ordinary differential equations from problem (2.7) is equivalent to the system of ordinary differential equations describing [1–3] the self-similar solutions $\mathbf{U}(t, r) = \mathbf{U}(r/t)$ of the system of equations of gas dynamics.

In the solution of the Cauchy problem (2.7) $s_{02}(\zeta) = s_0(0) = \text{const}$, while the functions $\sigma_{02}(\zeta)$, $u_{02}(\zeta)$ in general are not written in terms of quadratures [1]. However, in this system of ordinary differential equations the solutions and all their features are known [1]. Also, $\zeta = \zeta_* < 0$ and $\zeta = \zeta^* > 0$—the boundary points of the region in which an analytic solution of problem A exists—are known.

Here, in the case of the focusing of a compression wave the value $\zeta = \zeta_*$ (the straight line AO_0 in Fig. 2) corresponds to the sound characteristic AO from Fig. 1: $\sigma(\zeta_*, \chi) = 0$. In the case of the focusing of a rarefaction wave the value $\zeta = \zeta_*$ (AO_0 in Fig. 2) corresponds to the free boundary (AO in Fig. 1): $\sigma(\zeta_*, \chi) = 0$, $u(\zeta_*, \chi) = \text{const} = 1/\zeta_*$.

The value $\zeta = \zeta^*$ in both of the problems considered is greater than the value $\zeta = \zeta_1$: $\zeta^* > \zeta_1 > 0$, where $1/\zeta_1$ is the velocity of the reflected shock wave OC in the case of self-similar flows (the method of determining $\zeta = \zeta_1$ is described below).

In Figs 3 and 4 we show integral curves of problem A when $\nu = 2$ and $\gamma = 1.4$ (curves 1) and $\nu = 2$ and $\gamma = 3$ (curves 2).

In order to construct the remaining $\mathbf{U}_{k2}(\zeta)$, we must differentiate problem (2.3), (2.4) successively with respect to χ and put $\chi = 0$. We will thereby obtain linear systems of ordinary differential equations for $\mathbf{U}_{k2}(\zeta)$ with initial conditions $\mathbf{U}_{k2}(0) = \partial^k\mathbf{U}_0(\chi)/\partial\chi^k|_{\chi=0}$. When solving these Cauchy problems all the $\mathbf{U}_{k2}(\zeta)$ are uniquely defined. Series (2.6) thereby constructed is a reexpansion of series (2.5), which solves problem (2.3), (2.4). Since the systems of ordinary differential equations, from which $\mathbf{U}_{k2}(\zeta)$ $k \geqslant 1$ are determined, are linear, there are no singularities when $\zeta_* < \zeta < \zeta^*$ y $\mathbf{U}_{k2}(\zeta)$. We then prove, using the method described in [4, 5], that the points $\zeta = \zeta_*$ and $\zeta = \zeta^*$ are boundary points of the region of convergence of series (2.6) on the $O\zeta$ axis (and consequently, of series (2.5) also).

The solution of problem A is not constant, and it follows from this that in the space of the variables t, r in the region Ω_1, the vector functions $\mathbf{U}(t, r)$ take different values at the point ($t = 0$, $r = 0$) on

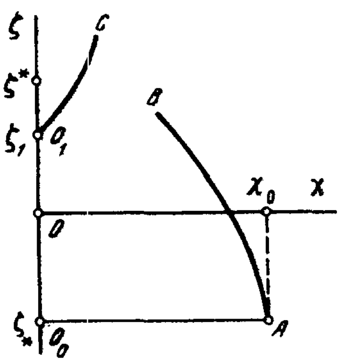

Fig. 2.

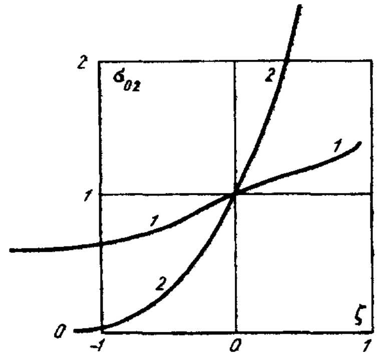

Fig. 3.

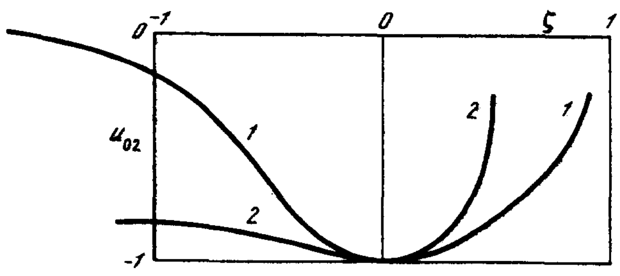

Fig. 4.

different straight lines t/r = const. Hence, the problem arises of choosing the values of the gas-dynamic parameters at the point ($t = 0, r = 0$) to calculate the velocity of motion of the reflected shock wave at the instant $t = 0$. We will assume that the trajectory of the reflected shock waves (the curve OC in Fig. 1) is given by the equation $r = \varphi(t)$, and the velocity of motion of the shock wave is then $D(t) = \varphi'(t)$. We must choose the value of ζ_1 so that $1/\zeta_1$ specifies the value of $D(0)$. For the function $D(t)$ we have the following relation from the Hugoniot conditions [16]

$$D = \frac{3-\gamma}{4} u_0 + \left[\frac{(\gamma+1)^2}{16} u_0^2 + c_0^2 \right]^{1/2} \tag{2.7}$$

when deriving which, the parameters of the gas in front of the shock wave are given the subscript zero, while the velocity of the gas behind the shock wave is assumed to be zero. Consequently, the required value of ζ_1 must satisfy the relation $1/\zeta_1 = y(\zeta_1)$, where $y(\zeta)$ is the right-hand side of (2.7), in which we must put $u_{02}(\zeta)$, $\sigma_{02}(\zeta)s_{02}(\zeta)$ instead of u_0 and c_0, respectively. Hence, for a specified solution of problem A the quantities ζ_1 are defined uniquely. This procedure is equivalent to determining the parameters of the gas behind the reflected shock wave OC when constructing solutions in the regions Ω_1 and Ω_2 in the class of self-similar flows [1–3].

From the specified analytic $r = \varphi(t)$ in the space of the variables ζ, χ, one can uniquely determine the analytic curve $\zeta = \varphi_1(\chi)$, passing through the point O_1 ($\zeta = \zeta_1$, $\chi = 0$) (the curve O_1C in Fig. 2). Then the parameters of the gas in front of the shock wave $r = \varphi(t)$—the solution of problem (2.3), (2.4) on the curve $\zeta = \varphi_1(\chi)$—are analytic functions of a single variable (of either r or t).

3. CONSTRUCTION OF THE SOLUTION IN THE REGION Ω_2 AND THE LAW OF MOTION OF THE REFLECTED SHOCK WAVE

Using the formulae

$$r = \varphi(\eta), \quad t = \theta + \eta \tag{3.1}$$

we replace r and t by the independent variables η, θ. The Jacobian of the conversion $J = \varphi'(\eta)$. Here the function $r = \varphi(t)$ is as yet unknown and specifies the trajectory of motion of the reflected shock wave. Using this substitution the $r = 0$ axis becomes the $\eta = 0$ axis and the shock wave—the curve OC—becomes another coordinate axis $\theta = 0$.

We will write the Hugoniot condition (16) on the shock wave (i.e. on the $\theta = 0$ axis) in the equivalent form for u, s, D in terms of $\mathbf{U}_1$ and l

$$u(\theta,\eta)|_{\theta=0} = \left[u_1 + \frac{2}{\gamma+1}\sigma_1 s_1\left(M - \frac{1}{M}\right)\right]_{\theta=0}$$

$$s(\theta,\eta)|_{\theta=0} = [s_1 R(M)]_{\theta=0}, \quad D|_{\theta=0} = (M\sigma_1 s_1 + u_1)|_{\theta=0}$$

$$R(M) = \{[(1+h)M^2 - h][(1-h) + hM^2]^{\gamma}\}^{1/2} M^{-\gamma}$$

$$\mathbf{U}_{00} = \mathbf{U}_1|_{\zeta=\zeta_1, \chi=0}, \quad l = \frac{2}{(\gamma-1)} s_{00}\sigma + \frac{2}{\gamma}\sigma_{00}s, \quad h = \frac{\gamma-1}{\gamma+1}$$

Here $\mathbf{U}$ is the solution in the region Ω_2, $\mathbf{U}_1$ is the solution in the region Ω_1, and the quantity M, as a function of σ_1, s_1 and 1, is found from the condition

$$\frac{2s_{00}}{\gamma-1}\{M[(1-h)+hM^2]^{-1/2}\}^{\gamma-1}\sigma_1 + \frac{2}{\gamma}\sigma_{00}s_1 R(M) = l$$

By the theorem of implicit functions, the function $M = M(\sigma_1, s_1, l)$ exists and is analytic in a certain neighbourhood of the point ($\sigma_1 = \sigma_{00}$, $s_1 = s_{00}$, $l = l^{00}$). Here the vector $\mathbf{U}^{00} = \mathbf{U}(\theta, \eta)|_{\theta=\eta=0}$ is known, since $D(0)$ and $\mathbf{U}_{00}$ are known.

We will rewrite the conditions for $u|_{\theta=0}$ in the form

$$u(\theta,\eta)|_{\theta=0} = [\beta l + q_1(\mathbf{U}_1, l)]_{\theta=0}; \quad \beta = \text{const} > 0, \quad \partial q_1 / \partial l|_{\theta=0,\eta=0} = 0$$

and we will denote the right-hand sides of the relations for $s|_{\theta=0}$, $D|_{\theta=0}$ by s^* and D^*, respectively.

Instead of the unknown functions $\mathbf{U}$ we will introduce the following new unknown functions

$$u' = u, \quad v = u - \beta l - q_1, \quad z = s - s^* \tag{3.2}$$

The Hugoniot conditions are then equivalent to the relations

$$v(\theta,\eta)|_{\theta=0} = 0, \; z(\theta,\eta)|_{\theta=0} = 0, \quad \varphi_\eta = D^*$$

The last of these is a differential equation for determining the unknown trajectory of motion of the shock wave.

For unknown functions μ' and φ, the initial data is specified on another coordinate axis

$$u'(\theta,\eta)|_{\eta=0} = 0, \quad \varphi(\eta)|_{\eta=0} = 0$$

The first of these conditions ensures that a zero value of the velocity of the gas is obtained in the region Ω_2 on the coordinate axis $r = 0$, and the second specifies the initial point of motion of the reflected shock wave OC ($t = 0$, $r = 0$).

In addition to the replacements (3.1) and (3.2) we apply an extension to the unknown functions μ' and v: $u'' = \varepsilon_1 u'$, $\upsilon' = \varepsilon_2\upsilon$. Here ε_1 and ε_2 are any two positive constants which satisfy the condition $\varepsilon_1/\varepsilon_2 = 2/(1 + a) \neq 0$, $a = (\beta - 1)/(\beta + 1)$, $|a| < 1$.

We solve the first two equations of the system thereby obtained from system (1.1) for the derivatives u_η, u_θ (the primes on u'' and v' are henceforth omitted). In addition, on the right-hand sides of the two equations obtained we convert the coefficients of u_θ, u_η as follows:

$$F = F_0 + F_1; \quad F_0 = F|_{O_2} = \text{const}, \quad F_1 = F - F_0; \quad F_1|_{O_2} = 0$$

where O_2 is a point with coordinates

$$\theta = 0, \quad \eta = 0, \quad u = 0, \quad v = 0, \quad z = 0, \quad \varphi = 0, \quad \sigma_1 = \sigma_{00}, \quad s_1 = s_{00}$$

Hence, to describe the flow in the region Ω_2 while satisfying the Hugoniot conditions on the line $r = \varphi(t)$ and the condition $u = 0$ on the line $r = 0$, we arrive at the following Cauchy problem with initial data, specified simultaneously on different coordinate axes

$$\begin{aligned}
u_\eta &= \frac{(1-a)(1-M_0^2)}{K_0} u_\theta + \frac{2M_0}{K_0} v_\eta - \nu \frac{(1-a)D^*}{K_0} \frac{u}{\varphi} + \Phi_1 \\
v_\theta &= \frac{2aM_0}{K_0} u_\theta + \frac{1-a}{K_0} v_\eta + \nu \frac{(1-a^2)D^*}{2K_0} \frac{u}{\varphi} + \Phi_2 \\
z_\theta &= E_1 u_\theta + E_2 v_\theta + E_3 u_\eta + E_4 v_\eta + \frac{u}{u - D^*} z_\eta + E_5 \\
\varphi_\eta &= D^* \\
u(\theta,\eta)|_{\eta=0} &= 0, \quad \varphi(\eta)|_{\eta=0} = 0, \quad v(\theta,\eta)|_{\theta=0} = 0, \quad z(\theta,\eta)|_{\theta=0} = 0
\end{aligned} \tag{3.3}$$

Here

$$M_0 = D(0) / c(\eta,\theta)|_{\theta=\eta=0}, \quad 0 < M_0 < 1$$
$$K_0 = (1-a) + M_0(1+a), \quad 0 < K_0 < 2$$

Φ_i, E_j $(1 \leqslant i \leqslant 2, 1 \leqslant j \leqslant 5)$ are specified functions of $\theta, \eta, u, \upsilon, \varphi, \sigma_1|_{\theta=0}, s_1|_{\theta=0}$, analytic in a certain neighbourhood of the point O_2, the derivatives u_θ, υ_η occur linearly in Φ_i, and the coefficients of these derivatives are zero at the point O_2. The specific form of Φ_i and E_j is not given here because of its complexity.

There is a singularity of the form u/φ in problem (3.3). Because of the presence of this singularity, problem (3.3) is not subject to the theorems from [10–14]. However, the corresponding theorem is proved when developing the method proposed in [14].

Theorem 2. If the solution $\mathbf{U}_1$ of problem (2.3), (2.4) is analytic in a certain neighbourhood of the point $(\zeta = \zeta_1, \chi = 0)$, then a unique analytic solution of problem (3.3) exists in a certain neighbourhood of the point $(t = 0, r = 0)$. A locally analytic law of motion of the shock wave OC—the curve $r = \varphi(t)$, on which the solutions of problem (2.3), (2.4) and problem (3.3) are related by the Hugoniot conditions, is also uniquely defined.

Without going into detail, we will simply give the main features of the proof of Theorem 2. The solution of problem (2.3) is constructed in the form of series

$$\mathbf{U}(\theta,\eta) = \sum_{k,l=0}^{\infty} \mathbf{U}_{k,l} \frac{\theta^k \eta^l}{k!\,l!}, \quad \varphi(\eta) = \sum_{n=0}^{\infty} \frac{\varphi_n}{n!} \tag{3.4}$$

The coefficients $\mathbf{U}_{k,l}$ $(k + l = n)$, φ_n are determined as follows. For $n = 0$ these coefficients are known from the initial conditions of problem (3.3). When $n = 1$ some of the coefficients are known from the initial conditions, while the remaining ones are determined by the values at the point O_2 of the right-hand sides of the equations of problem (3.3). If the coefficients are obtained for $k + 1 = 0, 1, \ldots, n$, then for $k + 1 = n + 1$ they are determined using this procedure. First, it follows from the initial conditions that $u_{0,n+1}$, $\upsilon_{n+1,0}$, $z_{0,n+1}$ are zeros. Second, we differentiate the system of equations from (3.3) k times with respect to η and once with respect to θ $(k = 0, 1, \ldots, n; k + l = n)$ and we assume $\eta = \theta = 0$. The result of differentiating the first two equations gives a system of linear algebraic equations for determining $u_{k,l}$ and $v_{k,l}$. If they are obtained, the result of differentiating the last two equations of system (3.3) will serve as the relations from which $z_{k,l}$, φ_{n+1} are determined explicitly. The system of linear equations for $u_{k,l}$ and $v_{k,l}$ has the form

$$v_{0,n+1} = D_1 v_{1,n} + Q_{0,n}$$
$$v_{1,n} = D_2 v_{2,n-1} + C_2 u_{1,n} + Q_{1,n-1}$$

$$
\begin{aligned}
&v_{k,n+1-k} = D_{k+1}v_{k+1,n-k} + C_{k+1}u_{k,n+1-k} + Q_{k,n-k},\\
&\qquad \ldots\\
&v_{n-1,2} = D_n v_{n,1} + C_n u_{n-1,2} + Q_{n-1,1}, \quad v_{n,1} = C_{n+1}u_{n,1} + Q_{n,0}\\
&u_{n+1,0} = A_{n+1}u_{n,1} + P_{n,0},\\
&u_{n,1} = A_n u_{n-1,2} + B_n v_{n,1} + P_{n-1,1},\\
&\qquad \ldots\\
&u_{k+1,n-k} = A_{k+1}u_{k,n+1-k} + B_{k+1}v_{k+1,n-k} + P_{k,n-k},\\
&u_{2,n-1} = A_2 u_{1,n} + B_2 v_{2,n-1} + P_{1,n-1}, \quad u_{1,n} = B_1 v_{1,n} + P_{0,n}
\end{aligned}
\tag{3.5}
$$

Here

$$
A_0 = \frac{(1-a)(1-M^2)}{K_0}, \quad B_0 = \frac{2M_0}{K_0}, \quad C_0 = \frac{2aM_0}{K_0}, \quad D_0 = \frac{1-a}{K_0}
$$

$$
f_0 = -\frac{\nu(1-a)}{K_0}, \quad g_0 = \frac{\nu(1-a^2)}{2K_0}
$$

$$
A_n = \frac{nA_0}{n-f_0} = \frac{n(1-a)(1-M_0^2)}{E_n}, \quad B_n = \frac{nB_0}{n-f_0} = \frac{2nM_0}{E_n}
$$

$$
C_n = C_0 + \frac{A_0 g_0}{n-f_0} = \frac{2naM_0 + \nu(1-a)(1+a-M_0+aM_0)/2}{E_n}
$$

$$
D_n = D_0 + \frac{B_0 g_0}{n-f_0} = \frac{(1-a)(n-\nu)}{E_n}, \quad E_n = nK_0 + \nu(1-a)
$$

where $P_{k,l}$ and $Q_{k,l}$ depend on $U_{k,l}$, φ_{k+l} when $k + l \leqslant n$.

The following inequalities hold

$$
\begin{aligned}
&D(0) > 0, \quad f_0 < 0, \quad C_0 < 1, \quad \gamma_0^2 > 4\alpha_0 \quad (B_0 + \alpha_0 - \beta_0)^2 > \alpha_0 > 0\\
&0 < B_n < 1, \quad (1-B_n)(1-C_n) > A_n D_n > 0 > -B_n(1-C_n) \quad n \in N\\
&(\alpha_0 = A_0 D_0, \quad \beta_0 = C_0 B_0, \quad \gamma_0 = 1 + \alpha_0 - \beta_0)
\end{aligned}
\tag{3.6}
$$

Satisfaction of inequalities (3.6) for $n = 0, 1, 2, \ldots$ ensures that the determinant Δ_n of system (3.5) is non-zero. It is found from the recurrent relations

$$
\begin{aligned}
&\Delta_0 = 1, \quad \delta_0 = 0, \quad \delta_n = 1 + A_n D_n B_{n-1}\delta_{n-1} / (B_n \Delta_{n-1})\\
&\Delta_n = 1 - C_{n+1} B_n \delta_n
\end{aligned}
$$

The solution of system (3.5) has the form

$$
\begin{aligned}
&v_{n,1} = \psi_{n+1,0}\\
&v_{k,n-k+1} = \sum_{i=k+1}^{n}\left[\left(\prod_{j=k+1}^{i}\frac{D_j}{\Delta_{j-1}}\right)\psi_{j+1,n-j}\right] + \psi_{k+1,n-k}, \quad k = n-1, \ldots, 0\\
&u_{n+1,0} = \chi_{n+1,0}\\
&u_{k,n+1-k} = B_k \Delta_k v_{k,n+1-k} + \chi_{k,n+1-k}, \quad k = n, \ldots, 1
\end{aligned}
$$

Here

$$
\chi_{1,n} = P_{0,n}
$$

$$\chi_{k+1,n-k}=\sum_{i=1}^{k}\left[\left(\prod_{j=i}^{k}\frac{A_{j+1}}{\Delta_j}\right)P_{i-1,n+1-i}\right]+P_{k,n-k}+$$

$$+\sum_{i=1}^{k}\left[\left(\prod_{j=i}^{k}\frac{A_{j+1}}{\Delta_j}\right)B_i\delta_iQ_{i,n-i}\right],\quad k=1,\ldots,n$$

$$\psi_{1,n}=Q_{0,n}/\Delta_0$$

$$\psi_{k+1,n-k}=(C_{k+1}\chi_{k,n+1-k}+Q_{k,n-k})/\Delta_k,\quad k=1,\ldots,n$$

The convergence of series (3.4) is proved by the majorant method.

The satisfaction of inequalities (3.6) ensures the existence of constants M_1, M_2, and q_*, which satisfy the inequalities

$$M_1\geqslant 1,\quad M_2\geqslant 1,\quad 0<q_*<1$$

$$\left(\prod_{i=n}^{k+n}\frac{|A_{i+1}|}{|\Delta_i|}\right)\leqslant M_1q_*^k,\quad \left(\prod_{i=n}^{k+n}\frac{|D_{i+1}|}{|\Delta_i|}\right)\leqslant M_1q_*^k$$

$$\frac{1}{|\Delta_k|}\leqslant M_2,\quad |B_n\delta_n|\leqslant M_2,\quad \frac{|C_{n+1}|}{|\Delta_n|}\leqslant M_2,\quad |A_k|\leqslant M_2$$

As a consequence of this, the Cauchy problem

$$\begin{aligned}
&W_t^*=M_3[1-(t+2U^*+W^*+Z^*)/\rho]^{-1}\\
&U_t^*=[(t+2U^*+W^*+Z^*)4U_t^*+1]W_t^*\\
&Z_t^*=[U^*W_t^*+4U_t^*+1]W_t^* \qquad\qquad (3.7)\\
&U^*(0)=W^*(0)=Z^*(0)=0,\quad \rho,M_3>0
\end{aligned}$$

majorizes the solution of problem (3.3). Here U^* majorizes u, υ ($U^*\gg u,\upsilon$); $W^*\gg\varphi$; $Z^*\gg z$; $t=\theta+\eta$.

By writing the differential system from problem (3.7) in normal form we obtain that the right-hand sides of this system will be analytic functions which majorize zero and, consequently, for problem (3.7) the Cauchy–Kovalevskii theorem holds. Hence, problem (3.7) has an analytic solution which majorizes series (3.4). This concludes the proof of Theorem 2.

Hence, the successive solution of problem (2.3), (2.4) and problem (3.3) describes the piecewise-analytic solution of the two problems considered on the focusing of a gas and the subsequent reflection of a shock wave with finite velocity from an axis or centre of symmetry.

Note. 1. Theorems 1 and 2 also hold for the case of a normal gas [15, 16] with an analytic equation of state.

2. If in problem (2.3), (2.4) the initial data are such that $\sigma_0(0)=0$, this leads to the occurrence of an additional singularity in problem (3.3). We have not investigated this case here.

3. Problem (2.3), (2.4) and problem (3.3), and also the methods described above for solving them can be used to devise numerical methods of constructing the corresponding gas flows.

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